

Coppersmith's Theorem XVII:

Coppersmith UNLEASHED

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Which theorem gives us all these **awesome** things?

1. **RSA key recovery**
2. **cryptanalysis of low-exponent stereotyped RSA**
3. **RSA-OAEP+**
4. **finding smooth integers in short intervals**

Coppersmith/Howgrave-Graham

Let

- ▶ $f(x) = x^d + f_{d-1}x^{d-1} + \dots + f_0$,
- ▶ N of unknown factorization,
- ▶ $0 < \beta \leq 1$.

Theorem

Can find all x_0 such that

$$\gcd(f(x_0), N) > N^\beta$$

$$|x_0| < N^{\beta^2/d}$$

in time polynomial in $\log N$ and d .

Proof idea

1. Form lattice from coefficients of $\{f(x)^i N^{k-i}\}_{i=0}^k$.
2. Find a short vector in the lattice.
3. Factor the polynomial you found.
4. Profit?

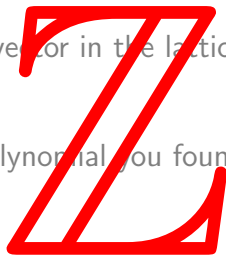
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Z

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[X]

Z

OK

Proof idea

O^s

$[f(x)]$

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2. Find a short vector in the lattice.

3. Factor the polynomial you found.

4. Profit?

OK

\mathbb{Z}

Polynomials!

Let

- ▶ $f(x, y) = y^d + f_{d-1}(x)y^{d-1} + \cdots + f_0(x)$,
- ▶ $N(x)$ of degree n ,
- ▶ $0 < \beta \leq 1$.

Theorem

Can find all $g(x)$ such that

$$\deg_x \gcd(f(x, g(x)), N(x)) \geq n\beta$$

$$\deg_x g(x) \leq n\beta^2/d$$

in time polynomial in n and d .

Polynomials!

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- ▶ $N(x)$ of degree n ,
- ▶ $0 < \beta \leq 1$.

Reed-Solomon list decoding!

Theorem

noisy polynomial interpolation!

$$\deg_x \gcd(f(x, g(x)), N(x)) \geq n\beta$$

$$\deg_x g(x) \leq n\beta^2/d$$

in time polynomial in n and d .

Number fields!

Let K n.f. of degree n , \mathcal{O}_K ring of integers,

- ▶ $f(x) = x^d + f_{d-1}x^{d-1} + \cdots + f_0 \in \mathcal{O}_K[x]$
- ▶ $I \subseteq \mathcal{O}_K$ an ideal,
- ▶ $0 < \beta \leq 1$.

Theorem

Can find all x_0 with $|x_0|_i < \lambda_i$ such that

$$N(\gcd(f(w)\mathcal{O}_K, I)) > N(I)^\beta$$

$$\prod_i \lambda_i < (2 + o(1))^{-n^2/2} N(I)^{\beta^2/d}$$

in time polynomial in n , $\log N(I)$, and d .

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$\mathfrak{a} \subseteq \mathcal{O}_K$ an ideal,

▶ $0 < \beta \leq 1$.

solving BDD in ideal lattices!

Theorem

Can find all x_0 with $|x_0|_i < \lambda_i$ such that

finding smooth elements in number fields!

$$\prod_i \lambda_i < (2 + o(1))^{-n^2/2} N(I)^{\beta^2/d}$$

in time polynomial in n , $\log N(I)$, and d .

Function fields!

Let K f.f., over curve \mathcal{X} , D divisor, $S \subseteq \mathcal{X}(\mathbb{F}_q)$,

- ▶ $f(x) = x^d + f_{d-1}x^{d-1} + \cdots + f_0 \in \mathcal{O}_S$,
- ▶ $I \subset \mathcal{O}_S$ an ideal,
- ▶ $0 < \beta \leq 1$.

Theorem

Can find all $x_0 \in (D)$ such that

$$N(\gcd(f(x_0)\mathcal{O}_S, I)) \geq N(I)^\beta$$

$$q^{\deg(D)} < N(I)^{\beta^2/d}$$

in probabilistic polynomial time.

Function fields!

Let K f.f., over curve \mathcal{X} , D divisor, $S \subseteq \mathcal{X}(\mathbb{F}_q)$,

- ▶ $f(x) = x^d + f_{d-1}x^{d-1} + \cdots + f_0 \in \mathcal{O}_S$,
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**list decoding of multi-point
algebraic geometry codes!**

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Can find all $x_0 \in (D)$ such that

$$N(\gcd(f(x_0)\mathcal{O}_S, I)) \geq N(I)^\beta$$

$$q^{\deg(D)} < N(I)^{\beta^2/d}$$

in probabilistic polynomial time.

Ideal forms of Coppersmith's theorem and Guruswami-Sudan list decoding

<http://arxiv.org/abs/1008.1284>